A MODIFICATION OF THE GAME OF NIM,

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1. The following arithmetical game is a modification of the game of „nim", described by C. L. Bouton in the *Annals of Mathematics*, 2nd series, vol. 3, p. 35 – 39.

The game is played by two persons. Two piles of counters are placed on a table, the number of each pile being arbitrary. The players play alternately and either take from one of the piles an arbitrary number of counters or from both piles an *equal* number. The player who takes up the last counter or counters, wins.

2. In accordance with C. L. Bouton I shall call a *safe combination* such a combination of two numbers as can be left safely on the table by one of the players, knowing that, if he do not make any mistake later on, the other player cannot win.

3. It is obvious, that the system of safe combinations has to satisfy the following conditions:

1°. that from a safe combination no other can be made by a move in accordance with the game-rules;

2°. that from every combination obtained from a safe combination by a move according to the game-rules, another safe combination can be made by the next move;

3°. that 0,0 is a safe combination.

4. The first condition consists of the following two:

that two safe combinations cannot have a number in common;

that the two numbers cannot have the same difference in two different safe combinations.

For the second the following more general condition may be substituted:

that from every combination not belonging to the set of safe combinations a safe combination can be obtained by a move according to the game-rules.

5. A set of combinations satisfying all these conditions is the following.

The first combination is 0,0. Of each following combination the smallest number is equal to the smallest number not occurring in one of the former combinations, while the difference of the numbers is greater by one than that of the preceding combination.

We thus find the following combinations:

0	0	9	15	19	31
1	2	11	18	21	34
3	5	12	20	22	36
4	7	14	23	24	39
6	10	16	26	etc.	
8	13	17	28		

That these combinations really satisfy all the conditions, is easily shown.

Hence they are the safe combinations.

6. If $E(x)$ denote the greatest integer not greater than x, then the combination

$$E\{\tfrac{1}{2}k(1+\sqrt{5})\} \quad , \quad E\{\tfrac{1}{2}k(3+\sqrt{5})\},$$

k being zero or a positive integer, is a safe combination, and we find all the safe combinations by successively substituting $k=0$, 1, 2, etc.

This theorem is proved as follows.

Since the difference of the two numbers $E\{\frac{1}{2}k(1+\sqrt{5})\}$ and $E\{\frac{1}{2}k(3+\sqrt{5})\}$ is k, it is evident, that by substituting $k=0$, 1, 2, etc. we obtain the series of differences of the combinations of numbers written down in § 5.

Hence it will be sufficient to prove, that this substitution produces once and not more than once any arbitrarily chosen positive integer.

Let n denote such an integer.

Let α and β be the smallest quantities which must be added to n to obtain multiples of $\frac{1}{2}(1+\sqrt{5})$ and of $\frac{1}{2}(3+\sqrt{5})$ respectively.

We then have

$$\alpha = \tfrac{1}{2}p(1+\sqrt{5}) - n \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$\beta = \tfrac{1}{2}q(3+\sqrt{5}) - n \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

p and q being integers, and

$$0 < \alpha < \tfrac{1}{2}(1+\sqrt{5}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$0 < \beta < \tfrac{1}{2}(3+\sqrt{5}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Multiplying (1) by $\frac{1}{2}(-1+\sqrt{5})$ and (2) by $\frac{1}{2}(3-\sqrt{5})$ and adding we find

$$\tfrac{1}{2}\alpha(-1+\sqrt{5}) + \tfrac{1}{2}\beta(3-\sqrt{5}) = p+q-n = \text{an integer.}$$

Multiplying (3) by $\frac{1}{2}(-1+\sqrt{5})$ and (4) by $\frac{1}{2}(3-\sqrt{5})$ and adding we find

$$0 < \tfrac{1}{2}\alpha(-1+\sqrt{5}) + \tfrac{1}{2}\beta(3-\sqrt{5}) < 2.$$

Hence the integer $p+q-n$ can be no other than unity, and we have

$$\tfrac{1}{2}\alpha(-1+\sqrt{5}) + \tfrac{1}{2}\beta(3-\sqrt{5}) = 1.$$

This equation is satisfied by $\alpha = \beta = 1$; but this solution has to be rejected, as the integer $n+1$ cannot be a multiple of $\frac{1}{2}(1+\sqrt{5})$ or of $\frac{1}{2}(3+\sqrt{5})$. Rejecting this solution we easily see, that one of the quantities α and β must be smaller and the other greater than unity.

It is evident, that, if $\alpha < 1$ and $\beta > 1$, we have $n = \mathrm{E}\{\frac{1}{2}p(1+\sqrt{5})\}$, and reversely, if $\alpha > 1$ and $\beta < 1$, $n = \mathrm{E}\{\frac{1}{2}q(3+\sqrt{5})\}$, and that n cannot be written in the form $\mathrm{E}\{\frac{1}{2}k(3+\sqrt{5})\}$ in the former, nor in the form $\mathrm{E}\{\frac{1}{2}k(1+\sqrt{5})\}$ in the latter case.

7. The following combinations of E-functions have properties similar to those of the combination considered in § 6:

$$\mathrm{E}(k\sqrt{2}) \text{ and } \mathrm{E}\{k(2+\sqrt{2})\};$$

$$\mathrm{E}\{\tfrac{1}{2}k(-1+\sqrt{13})\} \text{ and } \mathrm{E}\{\tfrac{1}{2}k(5+\sqrt{13})\}; \text{ etc.}$$

in general

$E\{\frac{1}{2}k(-a+2+\sqrt{a^2+4})\}$ and $E\{\frac{1}{2}k(a+2+\sqrt{a^2+4})\}$, a being a positive integer.

In all these combinations the substitution $k=1, 2, \ldots$ produces once and not more than once each positive integer. The series of differences however are $2, 4, 6, \ldots$; $3, 6, 9, \ldots$; in general a, $2a$, $3a, \ldots$

By their aid modified nim-games with suitably chosen game-rules might be constituted.

A still more general combination is the following:

$$E\left\{2-\frac{a}{2b}(b+2-\sqrt{b^2+4})+\tfrac{1}{2}k(-b+2+\sqrt{b^2+4})\right\}$$

and

$$E\left\{2+\frac{a}{2b}(b-2+\sqrt{b^2+4})+\tfrac{1}{2}k(b+2+\sqrt{b^2+4})\right\},$$

a and b being positive integers satisfying the condition $b \geqq a-1$.

In this combination the substitution $k=0, 1, 2 \ldots.$ produces once and not more than once each positive integer; the successive differences form the arithmetical series a, $a+b$, $a+2b$, ...

The proof of these properties may be left to the reader.